CIS5200: Machine Learning Spring 2023

Homework 0

Release Date: January 12, 2023 Due Date: January 20, 2023

This is the written part of HW0. There is also a programming part. The goal of written Homework 0 is to give you an idea of the level of mathematical knowledge and maturity expected in this course. You should have seen all this material before; the goal of this homework is to encourage you to go back to some of the material and refresh your memory.

- HW0 will count for 2% of the grade, and you will get full credit if you attempt all questions irrespective of whether the answer is correct, partially correct, or wrong.
- While we encourage collaboration for homeworks, in the specific case of HW0, you are on your own. It is a test of your level of readiness to take this course, so please work on it independently. Please use Ed Discussion only if you have clarifications about this homework.
- All homework solutions are required to be formatted using LaTeX. Please use the template here. This is a good resource to get yourself started, if you have not previosuly used LaTeX.
- You will submit your solution for the written part of HW0 as a single PDF file via Gradescope. The deadline is **11:59 PM ET**. Contact TAs on Ed if you face any issues.
- To get off the waitlist, complete HW0 and send your solutions to the Head TAs. Permission for remaining seats will be granted to those that complete HW0, as long as there is space availability.

Here is a list of resources to help brush up on the mathematical background:

- General Review - Mathematics for Machine Learning by Deisenroth, Marc Peter, A. Aldo Faisal, and Cheng Soon Ong, Cambridge University Press, 2020
- Linear Algebra Review
- Probability Review 1 and 2

Disclaimer: If you find HW0 to be very time consuming and extremely difficult, this course may not be right for you.

Note: Corrections and clarifications appear in red.

1 Written Questions

Q1 [Linear Algebra] Let A and B be real-valued square matrices of the same size. Which of the following identities are true? Give a proof or counterexample.

1. $AB = BA$
2. $tr(AB) = tr(BA)$
3. $(AB)^\top = A^\top B^\top$
4. $det(AB) = det(A) \cdot det(B)$

Q2 [Linear Algebra] Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 5 & 5 \end{bmatrix}$.

1. What is the rank of the matrix?
2. Give a (minimal) basis for the row span?

Q3 [Linear Algebra] Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$.

1. What are the eigenvalues and corresponding eigenvectors of A?
2. Is A a PSD (positive semidefinite) matrix?
3. What are the eigenvalues of $\lambda A + \gamma I$ as a function of λ, γ? Here I is the identity matrix.
4. What are the eigenvalues of A^2?

Q4 [Calculus] For column vector x ($n \times 1$ vector), answer the following questions:

1. Let $f(x) = w^\top x$ for column vector w ($n \times 1$ vector), compute $\nabla_x f(x)$.
2. Let $f(x) = (y - w^\top x)^2$ for column vector w ($n \times 1$ vector) and some scalar y, compute $\nabla_x f(x)$.
3. Let $f(x) = \log\big(1 + \exp(-y w^\top x)\big)$ for column vector w ($n \times 1$ vector) and some scalar y, compute $\nabla_x f(x)$.
4. Let $f(x) = x^\top A x$ for square matrix A ($n \times n$ matrix), compute $\nabla_x f(x)$. Simplify the expression for the case when A is symmetric.

Q5 [Geometry] Consider the hyperplane $w^\top x + b = 0$ for fixed vector $w \in \mathbb{R}^n$ and scalar b.

1. Under what conditions does the hyperplane pass through the origin?
2. What is the distance of any point x_0 from the hyperplane?

Q6 [Vector Norms] Let x be an n-dimensional vector. Denote by $\|x\|_p$, the p-th norm of x, that is, $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$. Which of the following are true identities? *No proof needed.*

1. $\|x\|_2 \leq \|x\|_1$
2. $\|x\|_1 \leq \sqrt{n}\|x\|_2$
3. $n\|x\|_\infty \leq \|x\|_1$
4. $\|x\|_0 \leq \|x\|_1$

Q7 [Convexity] Consider the two-dimensional function $f(x_1, x_2) = x_1^2 + x_2^4 - x_2^2$.

1. Is f a convex function?
2. What are the critical points of f, that is, points where the gradient is zero or undefined?
3. What is the minimum value that f attains?

Q8 [Probability] Suppose you want to go to the beach on Saturday. The ~~odds~~ chance of it being sunny on a Saturday are usually 10%. Since you want to go only if it's sunny, you decide to take help of a weather forecaster which has a 5% false positive rate (i.e. given that it is not sunny on Saturday, there's a 5% chance it says that it will be sunny) and a 15% false negative rate (i.e. given that it is sunny on Saturday, there's a 15% chance it says that it will not be sunny). Now suppose the weather forecaster says that it will be sunny on Saturday, what is the probability that you will actually enjoy a sunny day on the beach?

Q9 [Probability] Suppose z is distributed according to $N(\mu, \sigma^2)$, the Gaussian distribution with mean μ and standard deviation σ.

1. Find a, b such that $az + b \sim N(0, 1)$.
2. Compute $\mathbb{E}[z^2]$.
3. Suppose $\bar{z} \sim N(\bar{\mu}, \bar{\sigma}^2)$ independent of z, then what is the distribution of $z + \bar{z}$?

Q10 [Probability] Say you have a coin which has a probability p of showing up as heads.

1. What is the expected number of tosses you need to do before you see a heads?
2. Suppose you tossed the coin twice, and one of them showed up as heads. What is the probability that the other one was also facing heads?

2 Programming Questions

Use the link here to access the Google Colaboratory (Colab) for the programming. Be sure to make a copy by going to "File", and "Save a copy in Drive". This assignment uses the PennGrader system for students to receive immediate feedback.

Instructions for how to submit the programming component of HW 0 to Gradescope are included in the Colab notebook.